CHAOTIC ORBITS AND LONG TERM STABILITY: AN EXAMPLE FROM ASTEROIDS OF THE HILDA GROUP

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ABSTRACT

This paper calculates Lyapunov times, $T(\mathrm{L})$, for certain members of the Hilda minor planets, a group that librates at the 3:2 mean motion resonance with Jupiter. We find that many are definitely chaotic, but that the resulting escape times $T(c)$, obtained from a relation that we recently published [Lecar *et al.*, AJ, 104, 1230 (1992)] does not conflict with their being present today. On the other hand, we show that bodies with libration amplitudes 10 to 20 deg larger than the maximum currently found would have escaped during the lifetime of the solar system. We interpret this behavior as "observational" support for the relation between $T(\mathrm{L})$ and $T(c)$ that LFM inferred from numerical simulations. Consideration of $T(\mathrm{L})$'s for real and hypothetical objects with low proper eccentricities at and near the Hilda group lends further support.

1. INTRODUCTION

In a recent paper Lecar *et al.* (1992, hereafter LFM) found evidence for a statistical relation between the Lyapunov time, $t(\mathrm{L})$, [i.e., the inverse of the Lyapunov or characteristic exponent] and the time, $t(c)$, at which a body shows unmistakable signs of orbital instability, either by crossing the orbit of a massive planet or by encountering it within a few planetary radii. Here we shall use the latter definition for $t(c)$ and set the critical distance as 10 Jovian radii.

The support for this relation between orbital chaos and clear orbital instability was obtained from numerous computer experiments that resembled, in varying degrees, certain cases of solar system dynamics. According to LFM, the empirical relation between $t(c)$ and $t(\mathrm{L})$ is

$$t(c)/t(o)=A[t(\mathrm{L})/t(o)]^{\beta},$$

or, with $T(c)=t(c)/t(o)$ and $T(\mathrm{L})=t(\mathrm{L})/t(o)$:

$$T(c)=A[T(\mathrm{L})]^{\beta}. \tag{1}$$

In these equations, $\beta=1.8$, with a standard error of 0.1, $\log A=1.47\pm0.08$ and $t(o)$ is an appropriate normalizing orbital period, that of Jupiter, $P(\mathrm{J})$, in this case. Using these constants and taking the age of the solar system as 4.5×10^{9} yr, we find that objects for which $\log T(\mathrm{L})$, is less than 3.95 ($+0.28$; -0.25) should be absent or very rare. If Eq. (1) can be shown to have wide applicability, then the puzzling short Lyapunov times (10^{6}–10^{7} yr) obtained by Laskar (1989, 1990) as characterizing the inner planets would imply that their orbital instability will be manifest only in times much longer than the age of the solar system.

A search for supporting evidence for Eq. (1) among real objects in the solar system is the motivation for this paper. Our concern here will center on the Hilda group of minor planets, now with a membership of 53 numbered objects, all of which apparently librate stably at the 3:2 mean motion resonance ($a=0.763$) with Jupiter ($a=1.0$). These orbits are characterized by low inclinations, proper eccentricities, $e(p)$, that are all greater than 0.1, and by what is of special interest for this study: the amplitudes of their libration angles,

$$\phi=3\lambda(\mathrm{J})-2\lambda-\tilde{\omega}, \tag{2}$$

where λ and $\tilde{\omega}$ are mean longitude and longitude of pericenter and "J" refers to Jupiter, have a wide range, from near 0 to almost 90 degrees (cf. Table 1). Generally included as Hilda members, and added at the end of Table 1, are three objects [(334), (1256), and (4196)] with smaller semimajor axes ($\sim$0.749 vs 0.763) and low $e(p)$. Their distinctive motion, which does not include a libration of the form given by Eq. (2), has been discussed by Schubart (1991) and Franklin (1979).

The special characteristics of the Hildas enable us to check the validity of Eq. (1) by examining the following questions: (1) What are values of the Lyapunov time, $T(\mathrm{L})$, for the various mean amplitudes of libration, ϕ? (2) Are the $T(\mathrm{L})$'s for orbits with the largest observed $\phi=90$ deg consistent, via Eq. (1), with ages at least as old as that of the solar system, $T(\mathrm{ss})$? (3) Suppose we increase ϕ beyond the observed upper bound shown by two of the Hildas, i.e., >90 deg, what are the $T(\mathrm{L})$'s for such hypothetical bodies and are the resulting $T(c)$'s derived from Eq. (1) less than $T(\mathrm{ss})$? (4) Consider now the case of real and hypothetical objects with $e(p)\lesssim0.1$. The clear absence of real asteroids with $a\cong a(3{:}2)=0.763$ and $e(p)\lesssim0.1$ prompts the question whether bodies introduced with these elements will show small $T(\mathrm{L})$ and therefore $T(c)<T(\mathrm{ss})$. On the other hand, for the three well-established orbits (334) Chicago [$a=0.7468$, $e(p)=0.062$], (1256) Normannia (0.7495, 0.024), and (4196) Shuya (0.7517, 0.025), can we expect to find long $T(\mathrm{L})$ and $T(c)>T(\mathrm{ss})$? The check on Eq. (1) provided by the differing behavior of these two types of motion is particularly stringent because their semimajor axis difference is only 0.014

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TABLE 1. Characteristics of the Hilda planets. Third column gives the semimajor axis range associated with the libration whose amplitude is in col. 5. Data in cols. 4 and 5 have been either taken from Schubart (1991) or determined here. Final column gives the extent of the time period over which observations used for the MPC orbits have been made.

Planet Number	Mean a	Delta a	Proper Ecc.	Lib. Amp. (deg.)	Obs. Arc Yrs.
153	.76280	.00429	.172	19	130
190	.76308	.00746	.168	40	65
361	.76317	.00754	.206	45	85
499	.76332	.01061	.202	64	85
748	.76293	.00782	.168	43	70
958	.76308	.00830	.171	47	65
1038	.76326	.00942	.163	56	60
1162	.76285	.00898	.142	51	60
1180	.76305	.00721	.168	40	80
1202	.76299	.01088	.125	65	60
1212	.76299	.00477	.230	24	60
1268	.76270	.00863	.134	49	60
1269	.76264	.01428	.124	88	90
1345	.76295	.00533	.203	29	55
1439	.76305	.00834	.175	48	50
1512	.76305	.00809	.194	47	50
1529	.76328	.01115	.153	67	45
1578	.76322	.00942	.202	56	45
1746	.76310	.00525	.141	23	45
1748	.76353	.01103	.176	66	70
1754	.76329	.00772	..192	48	80
1877	.76300	.00604	.204	36	30
1902	.76319	.00329	.188	12	45
1911	.76278	.00529	.190	27	60
1941	.76343	.00832	.216	50	50
2067	.76308	.00648	.176	33	50
2246	.76280	.00684	.151	36	15
2312	.76264	.00738	.112	37	45
2483	.76295	.00421	.246	18	60
2624	.76258	.00677	.106	36	30
2760	.76284	.00800	.181	44	35
2959	.76332	.00728	.217	39	18
3134	.76320	.00609	.188	31	70
3202	.76274	.00988	.133	59	75
3254	.76286	.00794	.107	44	8
3290	.76271	.00561	.193	28	13
3415	.76325	.00263	.189	7	65
3514	.76315	.00940	.127	53	15
3557	.76326	.01013	.170	61	13
3561	.76264	.00460	.131	19	9
3571	.76263	.00784	.128	43	11
3577	.76299	.00818	.196	46	30
3655	.76361	.01263	.158	77	11
3694	.76310	.01010	.133	59	8
3843	.76268	.01188	.120	72	15
3923	.76308	.00482	.198	22	12
3990	.76306	.00663	.171	35	18
4230	.76286	.00480	.194	22	17
4255	.76302	.00522	.192	24	11
4317	.76295	.00654	.211	33	17
4446	.76309	.00516	.266	24	13
4495	.76278	.01445	.126	89	18
4757	.76251	.00581	.141	28	18
334	.74682	--	.062	--	90
1256	.74947	--	.024	--	60
4196	.75170	--	.025	--	15

[a(J) = 1.0] or 0.07 AU. The next section provides some answers to these four questions.

Our survey with $e(p) \lesssim 0.1$ and $a \simeq 0.763$ considered a set of only 15 orbits and is therefore incomplete. It probably should be extended to a completeness resembling the work of Murray (1986), who used a mapping scheme of limited accuracy to look for chaotic orbits at the 2:1 and 3:2 resonance. His paper does not provide Lyapunov times so that for this and other reasons there is no overlap between his study and this one.

2. RESULTS

This section discusses what can be deduced concerning orbital stability from the special features of the Hilda planets. To be concise, we shall omit a detailed summary of their motion and refer readers to a valuable set of papers by Schubart (see Schubart, 1991, and reference therein). The key feature that allows their apparent stable motion is the libration described by Eq. (2), which ensures that the conjunction of an asteroid and Jupiter occurs near the former's pericenter. Close approaches are thereby prevented. The first question we wish to investigate is how the Lyapunov time, T(L), depends upon the amplitude of libration, ϕ. To this end, we adopt a simple but appropriate model that we have used and discussed elsewhere (Franklin *et al.* 1989). This model includes Jupiter and Saturn and represents their motion by using the two dominant terms of secular theory. Thus their orbits are processing ellipses of varying eccentricities that range periodically between 0.012 and 0.085 (Saturn) and 0.028 and 0.061 (Jupiter).

As a group, the Hildas are characterized by low inclinations; in fact the four interesting members with librations greater than 70 deg all have proper inclinations less the 5 deg. Because of this fact, coupled with the low inclination of both Jupiter and Saturn, we have adopted with some confidence a planar model for all calculations reported here. To obtain T(L), we have relied upon an established technique (Soper *et al.* 1990) that looks for the presence of exponential growth in the longitude separation between two objects initially in identical orbits. We have repeatedly checked to assure that, when exponential behavior in longitude separation exists, it is also present (and with essentially the same time scale) in the full phase space.

We have calculated T(L)'s for more than half of the Hildas listed in Table 1, concentrating on those with libration amplitudes greater than 50 deg. The ability of the Minor Planet Center to integrate any asteroid for up to 400 years has provided us with a valuable library of initial conditions from which we have calculated T(L)'s at several epochs for a given object. Figures 1(a)–1(d), 2(a)–2(e), 3(a)–3(c), and 4 [also Figs. 9(a)–9(c) and 10(a)–10(c)] show the results of sample calculations from which T(L)'s were determined by a least-squares fit to the slope of the Log |longitude difference| versus time curves. Details are given by LFM. The first three cases, namely (1269) Rollandia, (499) Venusia, and (3655) Eupraksia are of special interest because of their large librations (cf. Table 1). Some of the curves in Fig. 2 that are used to explore their behavior are representative in the sense that when T(L) is very long, it cannot always be precisely determined even over 10^5 Jovian periods. This drawback is not a liability here because any log $T(\mathrm{L}) \gtrsim 4$ corresponds, via Eq. (1), to a collision time greater than the solar system age. Figure 4 for (3415) Danby, the Hilda with the smallest known libration of only 7 deg, is, by contrast, a case with a well-determined log T(L) that is greater than 5. The "a" cases in Figs. 1–3 use initial conditions of three real Hildas, while the remaining examples correspond to similar but fictitious objects with different libration amplitudes. These fictitious orbits were generated at a given epoch by rotating, normally by less than 30 deg, the minor planet's longitude of pericenter, $\tilde{\omega}$. Quite generally, changing $[\tilde{\omega}-\tilde{\omega}(\mathrm{J})]$ has a marked effect on ϕ, but a much lesser one on mean values of a and e. We can therefore regard the orbits in, for example, Figs. 1(b)–1(d) as reasonable extentions of the "real" asteroid shown in Fig. 1(a). Values of the libration amplitudes quoted on the figures have been

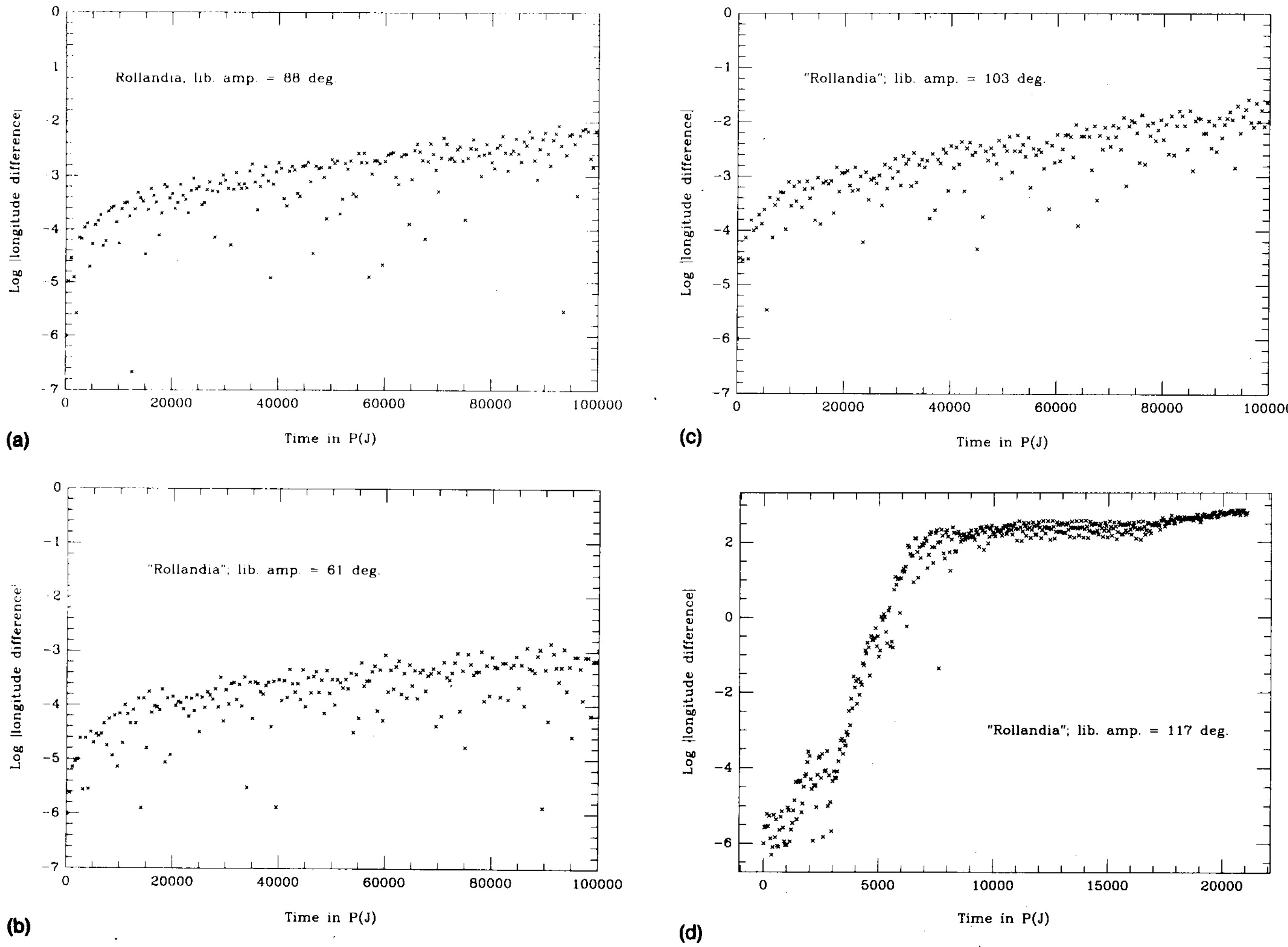


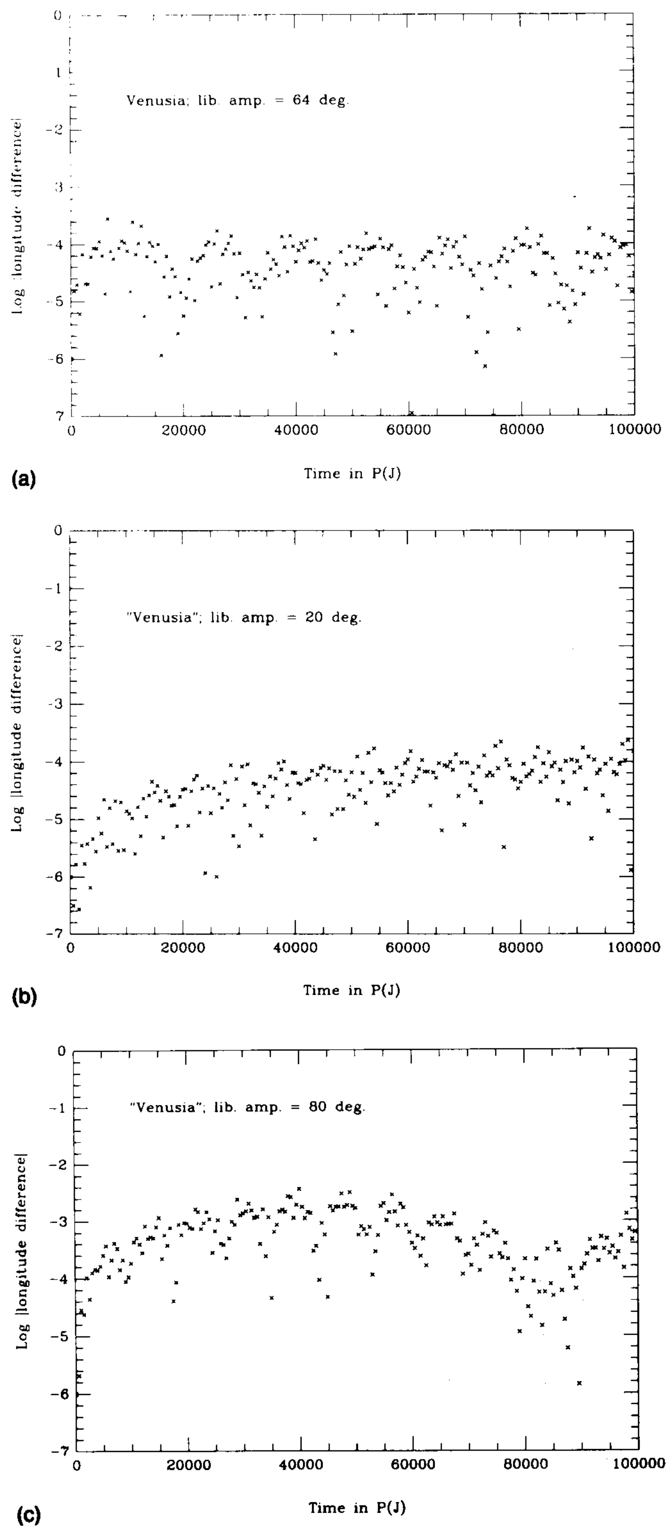
FIG. 1. (a)–(d) Figures 1–4, 9, and 10 plot the log of the longitude difference between two bodies in otherwise identical orbits as a function of time. Slopes of these curves determine the Lyapunov time. Figure 1(a) uses for initial conditions the orbital elements of (1259) Rollandia, which, together with (4495), has the largest libration. Other "Rollandia" type orbits with different amplitudes are generated by altering [$\tilde{\omega}-\tilde{\omega}(J)$] at $t=0$.

obtained by averaging over 60 libration periods, which are typically about 20 P(J) for the Hildas. Individual cycles show amplitudes that differ from the mean by ± 7 deg (499), ± 10 deg (1269), and ± 9 deg (3655).

We summarize results from these three cases in Table 2 and for the Hildas in general in Fig. 5. This figure shows there is a clear trend that links larger ϕ with shorter values of T(L). It is also clear that, though most of the real Hilda orbits are formally chaotic, the continued presence of such objects in the solar system after 4.5×10^9 yr [3.8×10^8 P(J)] leads to no contradiction if the relation expressed by Eq. (1) applies. In the same context, Fig. 5 argues that orbits with librations $\gtrsim 110$ deg are not likely to be found today.

Figure 6 shows twelve Hilda-type orbits that suffered close approaches to Jupiter. The dark solid line is the best fit to Eq. (1) as determined in LFM, and the dashed lines are 1 standard deviation boundaries. These Hilda-type orbits are consistent with values of A and β from LFM, but many more orbits should be integrated to times of order 10^7 P(J) to check whether the apparently smaller slope given by these data is statistically valid. [A time of (3–4) $\times 10^6$ P(J) is close to the effective operational limit of our integrations.] The asterisks in Table 2 denote three orbits with short Lyapunov times that were integrated to 2.5 $\times 10^6$ P(J), but which gave no signs of becoming unstable. Note that in two cases the minimum distance to Jupiter during the entire time remained greater than 0.275, or 1.43 AU. This distance is substantially greater than the ~ 1 AU limit that is generally accepted as dividing stable from unstable orbits. However, the three values of log T(L) at 2.5×10^6 P(J) are still less than 1.5 standard deviations from the LFM relation so that these cases cannot as yet be considered a problem.

Figure 7 displays values of the proper eccentricity, $e(p)$, and semimajor axis range, Δa, for the 53 permanently librating Hildas. [The linear relation between Δa and ϕ for all the Hildas is shown in Fig. 8 and can be approximately predicted by first-order theory, cf. Greenberg (1973).] Figure 7 emphasizes the absence of Hildas with $a=0.763$ and $e(p)\lesssim 0.1$. Figures 9(a)–9(c) show the characteristic behavior of log |longitude difference| versus time for 3 of 15 cases with initial conditions in this region. The associated log T(L)'s are 3.19, 3.64, and 4.01 P(J) and the indicated lengthening of T(L) with increasing $e(p)$ is typical of the 15 hypothetical orbits with mean semimajor axes $\simeq 0.763$.



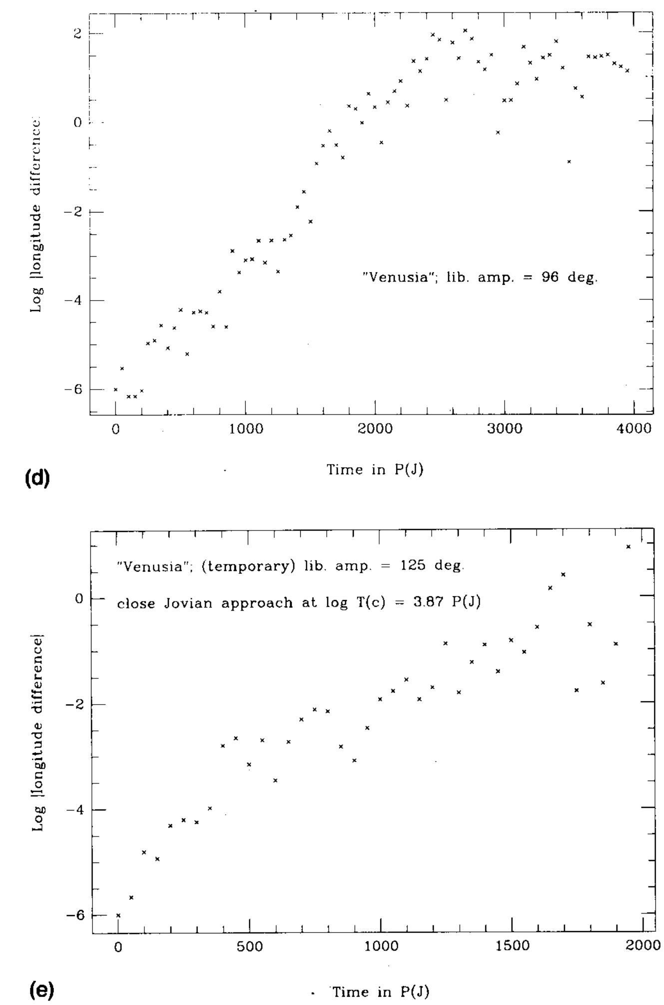


FIG. 2. (a)–(e) Growth of the longitude difference for (499) Venusia. Horizontal scale is always measured in Jovian periods, $P(\mathrm{J})=11.86$ yr. Orbit shown in (e) approached Jupiter to within 10 radii at $\log T(c)=3.87\ P(\mathrm{J})$.

The example shown in Fig. 9(a) collided with Jupiter during an extended integration at $\log T(c)=6.55\ P(\mathrm{J})$, while the other two had approached the planet to 1.12 and 1.17 AU during integrations spanning $2\times10^6\ P(\mathrm{J})$. It seems reasonable to conclude that orbits with $e(p)$ smaller than about 0.1, if once present in this region, have a statistical likelihood of being ejected during the lifetime of the solar system.

A qualitatively different behavior—slight but significant—occurs when we examine orbits with similar or smaller $e(p)$'s, but with $a\simeq0.750$, i.e., orbits like the three numbered asteroids mentioned earlier. Consider first (334) Chicago. With the largest $e(p)$ (0.062) of the three, and with $\log T(\mathrm{L})=4.46\ P(\mathrm{J})$, it is, as Fig. 10(a) indicates, the most stable. But the crosses derived from the actual orbital elements and the filled dots obtained by rotating the apsidal line relative to Jupiter's by only 4 deg indicate that the structure of phase space near such orbits is complex. An examination of Chicago's motion demonstrates its unusual character. As distinct from the regular Hildas, where conjunctions with Jupiter librate about pericenter, in Chicago's case they occur all around the orbit, including at or

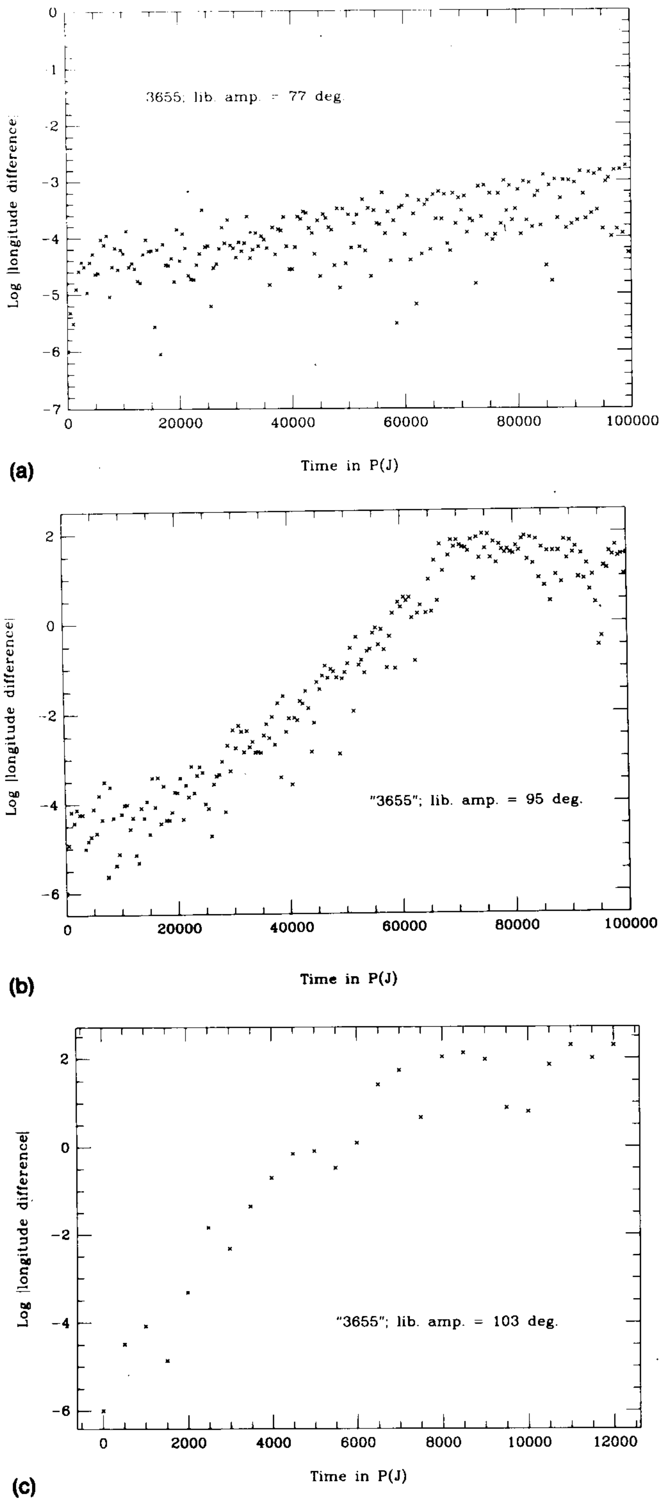


FIG. 3. (a)–(c) The case of (3655) Eupraksia. Orbits with libration amplitudes differing from the actual case are placed in quotes.

near apocenter. At the same time, the apsidal line difference, $[\tilde{\omega}-\tilde{\omega}(J)]$, normally can make full rotations. Since Chicago's eccentricity, e, can be as large as 0.12, one might expect the possibility of Jovian encounters well below 1 AU. Such close encounters are prevented by what is easiest to visualize as a relation between the period of the resonant term [which causes a periodic variation of amplitude $e(r)=0.021$ in e] and the synodic period of Chicago and Jupiter. This relation (details in Franklin 1979) ensures that conjunctions at Chicago's apocenter occur when the phase of $e(r)$ reduces e to a minimum. Conversely, maximum e corresponds to conjunction at pericenter.

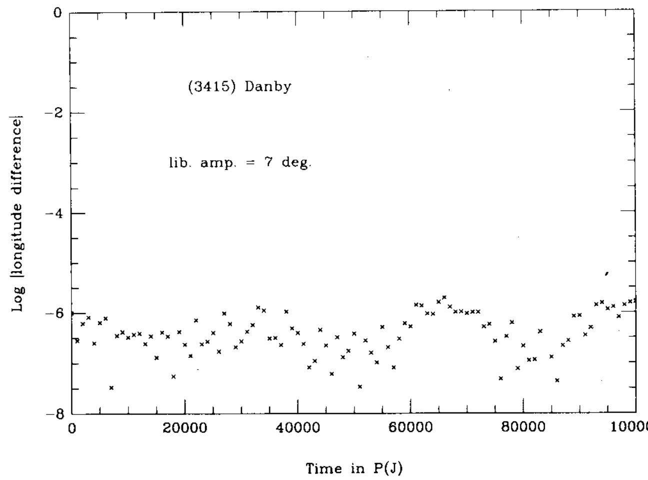


FIG. 4. Longitude difference vs time for (3415) Danby, the Hilda member with the smallest observed libration amplitude.

Figures 10(b)–10(c) plot the longitude versus time curves for the two very small $e(p)$ (0.025) asteroids (1256) and (4196). For both cases $\log T(\mathrm{L})=3.96$ so that formally their existence at the present time is not surprising, but their long time stability is doubtful. In the case of (4196), a 4 deg shift in $[\tilde{\omega}-\tilde{\omega}(J)]$ leads to a more stable condition, but it makes no difference for (1256). The apparently complex nature of the stable/unstable domain near these orbits and our present ignorance of why orbits with small $e(p)$ are unstable argues that a more detailed survey, probably in 3 dimensions, would be appropriate.

TABLE 2. Parameters for three real and extended (definition in text) orbits. Underlined cases refer to the actual orbit. An asterisk (*) denotes an orbit integrated to 2.5×10^6 Jovian periods, $P(J)$, for which no close encounter with Jupiter occurred. The solar system age in $P(J)$ is $\log T(ss)=8.58$.

Mean lib. Amplitude degs.	Lyapunov and collision times [Eq.1] in P(J) log T(L)	log T(c)	Min. Jovian dist. [J=1.0]	Observed log T(c)
		(a) (499) Venusia		
20	4.64	9.8	0.401	
64	>5	>10.5	0.359	
80	>4.50	> 9.6	0.333	
96	2.17	5.4	0.286	*
125	2.03	5.1	-	3.87
		(b) (1259) Rollandia		
61	4.56	9.7	0.287	
88	4.45	9.5	0.265	
103	4.31	9.2	0.258	
117	2.51	6.0	0.206	*
		(c) (3655) Eupraksia		
77	4.50	9.6	0.301	
95	3.64	8.0	0.296	
103	2.61	6.2	0.275	*

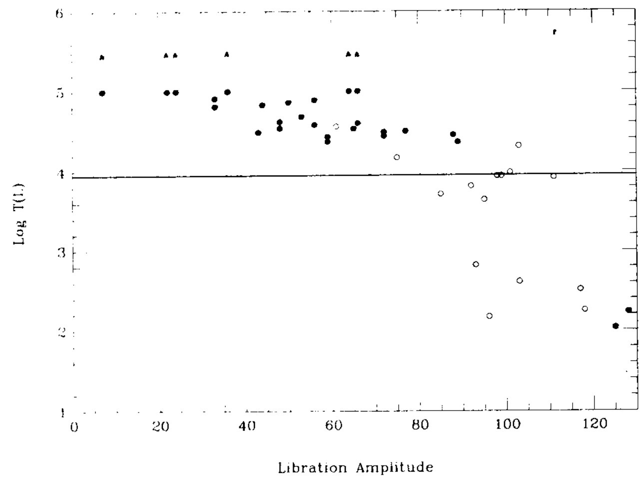


FIG. 5. Filled dots give the libration amplitude (deg) vs log [Lyapunov time] for 25 orbits of the Hilda group. The 6 points attached to arrows indicate lower limits. Open circles refer to real Hilda orbits except that just one element [$\tilde{\omega}-\tilde{\omega}(J)$] has been changed in order to alter the libration amplitude. Crosses within the open circles mark two cases (of five examined) that collided with Jupiter. The solid horizontal line at log $T(L)=3.95$ denotes the boundary below which, according to Eq. (1), orbits are likely to have escaped during the solar system age. Dotted lines are 1σ limits.

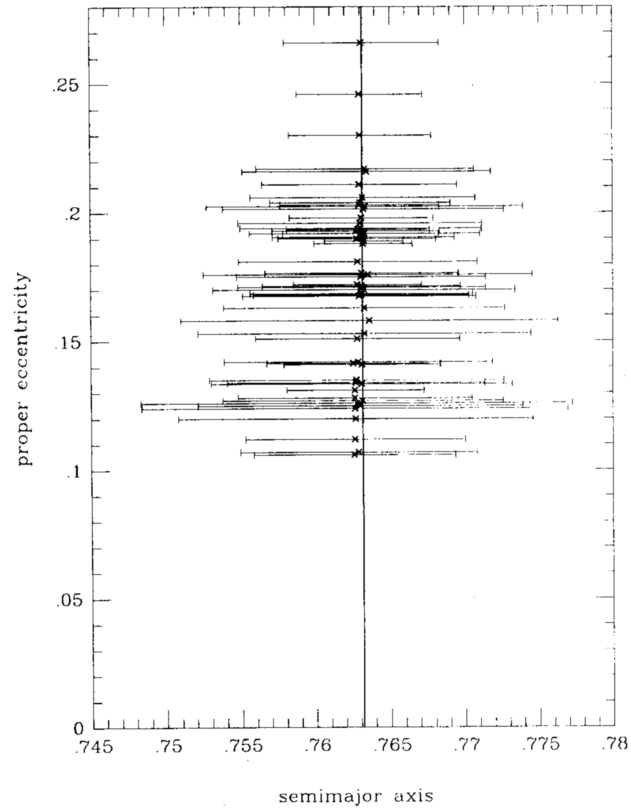


FIG. 7. Proper eccentricity, $e(p)$, vs semimajor axis range for 53 Hildas, where the former have been taken from Schubart (1991) or determined here. Vertical line is the position of the 3:2 resonance (with $\dot{\tilde{\omega}}=0$), at $a=0.76314$. The trend toward smaller a for low $e(p)$ is a consequence of more rapid apsidal motion as eccentricity falls. Note the absence of any Hildas with $e(p)<0.1$.

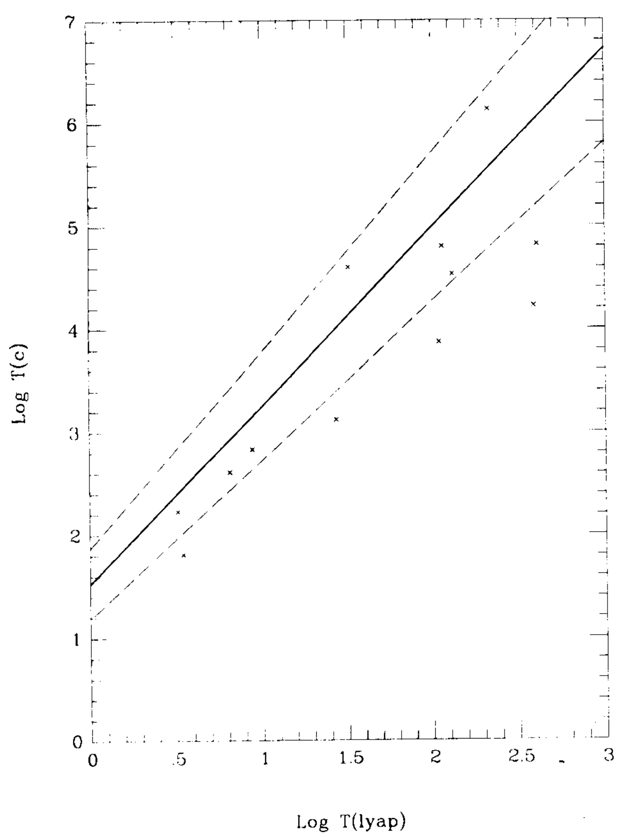


FIG. 6. Collision time vs Lyapunov time, where the former means that an object has approached Jupiter within 10 radii, for 12 Hilda-type orbits. Solid line, and the dotted lines giving 1σ limits, are taken from LFM, and also given by Eq. (1).

3. CONCLUSIONS

This paper has examined the long-term stability of the Hilda minor planets by calculating Lyapunov times, $T(L)$, for many members. The paper is not directed toward a discussion of the details of their motion (which have been examined by Schubart 1991) nor a mapping of the reso-

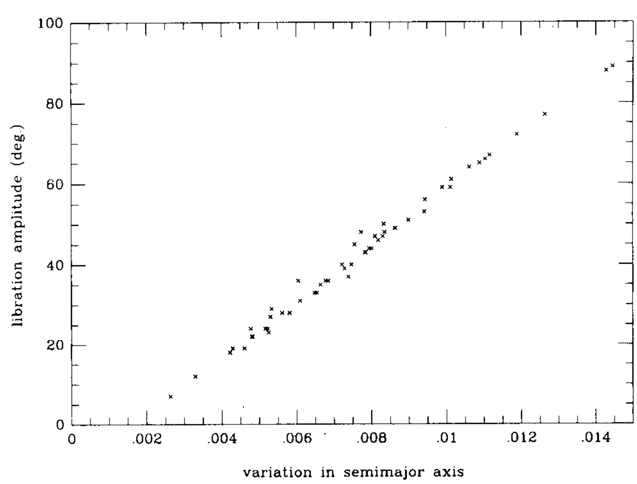


FIG. 8. The observed relation between the amplitude of libration and the corresponding variation in semimajor axis for 53 Hildas.

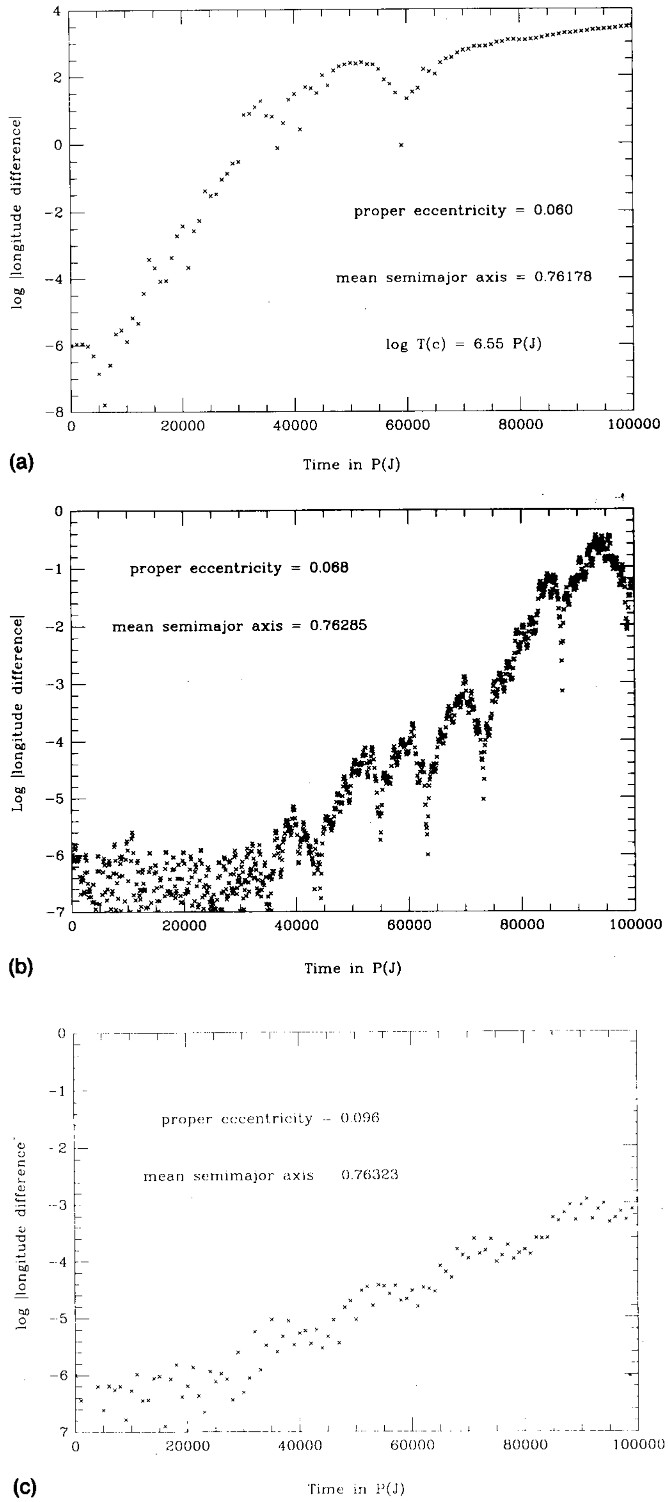


FIG. 9. (a)–(c) Three examples of quite rapid exponential growth in longitude separation for hypothetical orbits of small proper eccentricity. Case with the lowest $e(p)$ collided with Jupiter at the indicated time, while the example with the largest $e(p)$ gives $\log T(\mathrm{L})=4.01$, which is essentially at the escape boundary after 4.5×10^9 yr (cf. Fig. 5).

nant region (see Murray 1986). Rather, our concern is to establish whether values of $T(\mathrm{L})$ for all real and related Hilda members do support the relation introduced by LFM (1992) between the Lyapunov time, $T(\mathrm{L})$, and the collision time, $T(c)$. This support seems quite well established inasmuch as we show that: (a) $T(\mathrm{L})$'s for all the

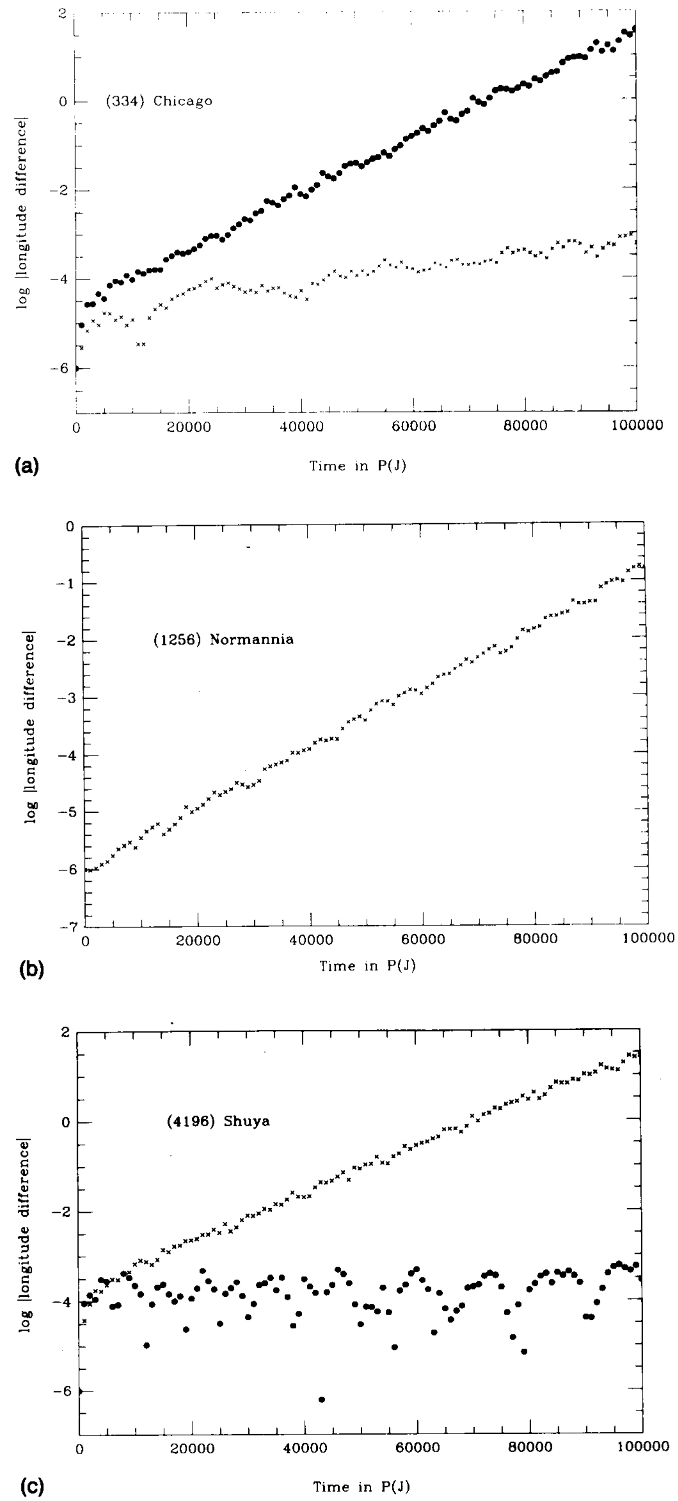


FIG. 10. (a)–(c) Longitude difference vs time for the three "associate members" of the Hilda group that have smaller semimajor axes and that do not librate permanently. Crosses refer to results derived from actual orbital elements and filled dots when $[\tilde{\omega}-\tilde{\omega}(\mathrm{J})]$ was increased at $t=0$ by 4 deg for (334) (apparently making it less stable) and decreased by 4 deg for (4196). Results for (1256) remained essentially identical for such changes in $\tilde{\omega}$.

Hildas are consistent with lifetimes greater than the age of the solar system; (b) if we increase the amplitude of libration by 10 to 20 deg (cf. Fig. 5) above the largest observed values, then the resulting $T(\mathrm{L})$'s imply $T(c)$'s that are less than the age of the solar system, and (c) objects placed in the 3:2 resonance with low proper eccentricities (where no real asteroids are found) all exhibit short values of $T(\mathrm{L})$. Evidence favoring the LFM relation is, however, somewhat tempered by our inability (due to the $>10^7$ yr integrations required) to follow all of the individual orbits with relatively short $T(\mathrm{L})$'s until a close encounter with Jupiter occurs. Because we find values of log $T(\mathrm{L})$ of about 4.5 [in $P(\mathrm{J})$] among the Hildas, we would argue that a continued, gradual decline in their population will occur over the course of time.

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